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2007 J. Phys. A: Math. Theor. 40 9361

(http://iopscience.iop.org/1751-8121/40/31/013)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9361–9369 doi:10.1088/1751-8113/40/31/013

An approximate symbolic solution for convective instability flows in vertical cylindrical tubes

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Received 10 December 2006, in final form 23 May 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9361

Abstract

The convective flow in vertical cylindrical tubes is investigated and a new formula for its velocity is derived. The Ostroumov problem is briefly discussed, and the relevant fourth-order ordinary differential equation referring to this problem is solved directly in its complete form and within a frame of an allowed simplification, as well. The result obtained for the velocity function is in good qualitative agreement with earlier simulation calculations.

PACS number: 44.30+v

1. Introduction

As is obvious, the investigation of convection phenomena represents nowadays one of the important research topics in both the fundamental hydrodynamics and related engineering areas [1–3]. The convection phenomena appear sometimes in flow patterns of symmetric forms (e.g. in case of the Rayleigh–Bènard instability [4, 5]). They can be treated adequately within the framework of general fluctuation theory of phase transitions [6]. These types of phenomena set up attractive examples of convection flows, which are unsolved problems of hydrodynamics. There are some other appearances of the convective flow patterns, creating an equally interesting point of view of fundamental research, as well as from engineering applications of them. One of them is a problem studied by Ostroumov [7, 3] related to the examination of convection instability threshold in vertical tubes. The mathematical modelling of these phenomena is an actual research topic [2, 8, 9] having some direct technical applications too. Among them, modelling of flow characteristics of different technical equipments [8] represents demonstrative examples. Furthermore, study of influence of rough surface (through turbulent boundary layer phenomena) on a heat flux at the turbulent convective flow was possible [9] by use of particular-type special functions, namely the Lambert-W functions [10] and calculations by using a symbolic computer algebra system MAPLE.

Therefore, applications of advanced software packages make possible realization of symbolic computational model experiments so as to understand the character of such flows. In the present work, a new approach is shown in order to approximately determine the functions of velocity and temperature distribution in vertical tubes having cylindrical symmetry and insulating walls.

2. General formalism of the convection phenomena

According to the classical descriptions of free convection [3] it is supposed that changes of fluid density ρ caused by temperature variations lead to the appearance of convective driving forces, while fluid density changes due to the variations of pressure p are negligible (i.e. the very high vertical fluid columns are excluded from consideration). The Boussinesq system of partial differential equations (PDEs) is relevant for describing the velocity function $\vec{\mathrm{v}}$ of stationary convection:

$$\begin{aligned}(\vec{\mathrm{v}}\cdot\nabla)\vec{\mathrm{v}} &= -\nabla\frac{p}{\rho} - \vec{g}\cdot\beta\cdot T + \frac{\eta}{\rho}\nabla^2\vec{\mathrm{v}},\\ \vec{\mathrm{v}}\cdot\nabla T &= \chi\cdot\nabla^2 T, \qquad \nabla\cdot\vec{\mathrm{v}} = 0,\end{aligned} \tag{1}$$

where β denotes compressibility, η is the dynamic viscosity coefficient, $\vec{g}$ is the gravitational acceleration, T is the temperature and χ denotes the heat propagation coefficient. This well-known system of PDEs can be derived directly from the basic balance equations of non-equilibrium thermodynamics [3], and transformed into a concise form after general discussion of the ordinary problem of convective instabilities mentioned above. This transformation leads to a fourth-order linear partial differential equation, i.e.,

$$\Delta^2 v = \frac{Ra}{R^4}v, \qquad \left(Ra \equiv \frac{CR^4 g\beta\rho}{\chi\eta}, \Delta \equiv \nabla^2\right), \tag{2}$$

where Ra is the Rayleigh number (this important parameter is playing a decisive role in description of convective phenomena). It can be represented as a product of the Grashof, and Prandtl numbers, i.e. $Ra = Gr \cdot Pr$, and the latter is also a well-known dimensionless quantity: $Pr = \frac{\eta}{\chi\cdot\rho}$, R is the radius of the cylinder, C is the vertical temperature gradient of a constant value, while the Laplacian operator $\Delta \equiv \nabla^2$ (as well as its quadratic form) is usually given in cylindrical coordinates at solving of this problem. Another important relationship emanating from the same classical theoretical description of the convection problem is the following:

$$\chi\Delta\tau = -Cv, \tag{3}$$

where τ denotes the temperature perturbation, which may cause convection instability. Although equations (2) and (3) have been studied many times [3, 7], we thought that it would be useful to examine this problem again and remodel its main features in a novel manner basing our work on the powerful, new and symbolic calculation techniques.

The calculations were performed entirely by use of the MAPLE [11] and MATHEMATICA [12] computer software packages, and it is shown that neither the application of Bessel functions (with both real and imaginary arguments) [3], nor the use of the Fourier series expansion technique (with respect to the spanwise aspect ratio—used e.g. in [8]) are necessary for studying the velocity fields in such tubes described basically by the fourth-order partial differential equation (2).

3. Results and discussion

Firstly, we solved the radial part of PDE (2) directly. Taking into account cylindrical symmetry of the problem, for calculating the velocity function in the cross section of the vertical tube, it is natural to use polar coordinates (r, φ)instead of rectangular ones (x, y). Here r denotes the radial distance from the axis of cylinder, i.e. $0 \leqslant r \leqslant R$. By fixing the value of the angle φ (and modifying therefore the square of the planar Laplacian operator expressed in polar coordinates), we arrive at the following simplified ordinary differential equation (ODE)) originated from PDE (2):

$$\frac{\mathrm{d}^4}{\mathrm{d}r^4}v + \frac{2}{r}\cdot\frac{\mathrm{d}^3}{\mathrm{d}r^3}v - \frac{1}{r^2}\cdot\frac{\mathrm{d}^2}{\mathrm{d}r^2}v + \frac{1}{r^3}\cdot\frac{\mathrm{d}}{\mathrm{d}r}v = av, \tag{4}$$

whose solution is (by application of the MATHEMATICA 5.2 symbolic computer program)

$$\begin{aligned} v(r) = {}& \frac{\mathrm{i}}{8}C_1\cdot\{I_0(\sqrt[4]{a}r) - J_0(\sqrt[4]{a}r)\} + \frac{1}{2}C_2\{I_0(\sqrt[4]{a}r) + J_0(\sqrt[4]{a}r)\} \\ & + C_3\cdot G^{2,0}_{0,4}\left[\frac{ar^4}{256}\middle| 0, 0, \frac{1}{2}, \frac{1}{2}\right] + C_4\cdot G^{2,0}_{0,4}\left[\frac{ar^4}{256}\middle| \frac{1}{2}, \frac{1}{2}, 0, 0\right], \end{aligned} \tag{5}$$

where C_i $(i = 1, 2, 3, 4)$ denote integration constants, J_n and I_n are Bessel functions of nth order with real, and imaginary arguments, respectively (and interconnected via well-known relationship $I_n(z) = J_n(\mathrm{i}z)\times \mathrm{e}^{-\mathrm{i}\frac{n\pi}{2}}$, z is a real or complex argument), while Meijer's G-functions are defined [13] by the following expression:

$$\begin{aligned} & G^{mn}_{pq}\left(x\middle|\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q\end{matrix}\right) \\ & \quad = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_L}\frac{\Pi^m_{j=1}\Gamma(b_j+s)\Pi^n_{j=1}\Gamma(1-a_j-s)}{\Pi^p_{j=n+1}\Gamma(a_j+s)\Pi^q_{j=m+1}\Gamma(1-b_j-s)}x^{-s}\,\mathrm{d}s, \end{aligned} \tag{6a}$$

where $\Gamma(s)$ denote the gamma function and the integration contour lies between the poles of $\Gamma(1-a_i-s)$ and $\Gamma(b_i+s)$. Therefore, the Meijer functions in (5) specialize to

$$G^{2,0}_{0,4}\left(x\middle|0, 0, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_L}\frac{\Gamma(s)^2}{\Gamma\left(\frac{1}{2}-s\right)^2}x^{-s}\,\mathrm{d}s, \tag{6b}$$

and to

$$G^{2,0}_{0,4}\left(x\middle|\frac{1}{2}, \frac{1}{2}, 0, 0\right) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_L}\frac{\Gamma\left(s+\frac{1}{2}\right)^2}{\Gamma(1-s)^2}x^{-s}\,\mathrm{d}s, \tag{6c}$$

respectively.

Similarly, application of the MAPLE 10 software package gave us the following result for the same problem:

$$v(r) = C_1'\cdot J_0(\sqrt[4]{a}r) + C_2'\cdot Y_0(\sqrt[4]{a}r) + C_3'\cdot J_0(\mathrm{i}\sqrt[4]{a}r) + C_4'\cdot Y_0(\mathrm{i}\sqrt[4]{a}r), \tag{7}$$

where $a \equiv \frac{Ra}{R^4}$, C_1', C_2', C_3', C_4' are again integration constants and functions $Y_\nu(z)$ are the Bessel functions of second kind, sometimes also called Neumann functions (ν is a real number, which may take both integer and non-integer values). It is an important feature of Neumann functions that they have singularity at the zero value of argument. Solutions (5) and (7) should be equivalent. Then, we present graphically our calculation results (figure 1).

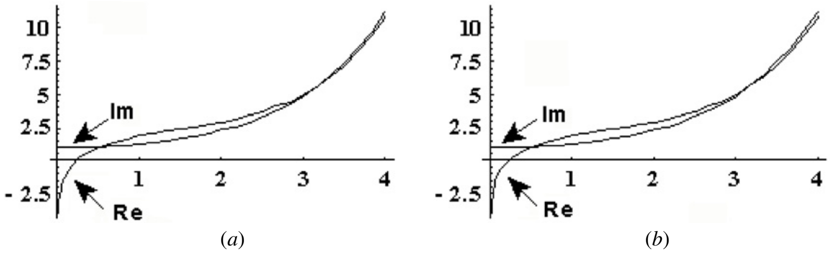

Figure 1. Numerical presentation of real (*Re*) and imaginary (*Im*) parts of solutions of ODE (4) by MATHEMATICA 5.2 (*a*) and MAPLE 10 (*b*) in the interval (0.01;4); for the sake of simplicity all integration constants and *a* are taken to have unit value.

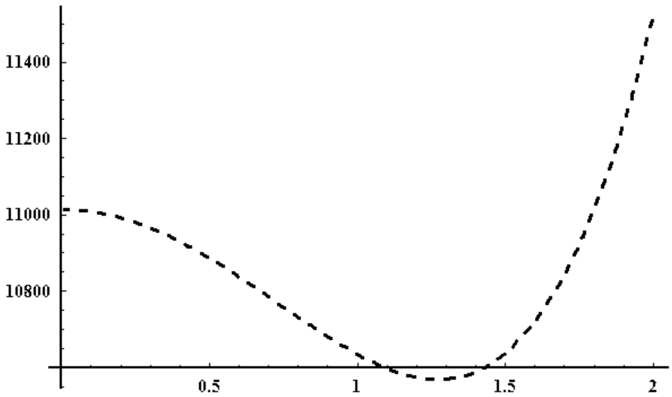

Figure 2. Graphical presentation of the real part of the exact solution of the ODE (4) realized by the numerical method of MATHEMATICA 5.2 ($r = 0.01..2$). (The initial conditions were the next: $f(0.01) = 11\,014.3$; $f'(0.01) = -11.0137$; $f''(0.01) = -1101.36$; $f''''(0.01) = 2.9938$).

The curves above illustrate that the exact solutions obtained by two software packages are identical.

The curves have shown in figures 2–5 are calculated by use of data indicated at the bottom of the figures 2 and 3. All of them are related to velocity functions of the convective flow in vertical cylindrical tubes.

It is obvious that the boundary conditions considered in this paper (i.e. the Dirichlet- and Neumann-type conditions corresponding to vertical cylindrical tubes with insulating walls and to be discussed below) can hardly be analysed by use of the general solution formulae (5), (7). Particularly, the gradient of the temperature function (which can be calculated by use of (3) via one-time integration of the velocity function) is expressed analytically (solution is obtained by MAPLE 10) using Bessel-, and Struve-type special functions. This calculation result is presented in the appendix (equation (A.1)).

In order to avoid such labouroursome calculations, we simplify the ODE (4) using following assumptions:

- In the region the nearby axis of the vertical cylinder, the slope of the velocity curve is very small, leading to small values of the first, second and third derivatives of this function. Then, taking large enough values for r in the proximity of the cylinder axis, the terms $\frac{2}{r} \cdot v'''(r)$, $\frac{1}{r^2}v''(r)$, $\frac{1}{r^3}v'(r)$may tend separately to value zero.

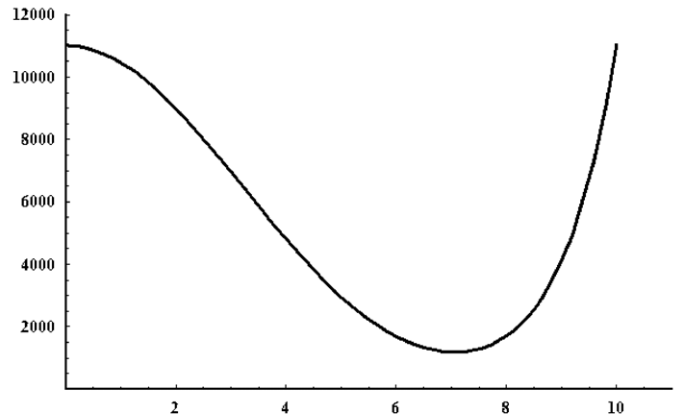

Figure 3. Vertical component of the velocity function in a vertical cylindrical tube (for $K = 100$, $K_1 = 1, K_2 = 1, K_3 = 1, A = 1, 0{,}01 \leqslant r \leqslant 10$).

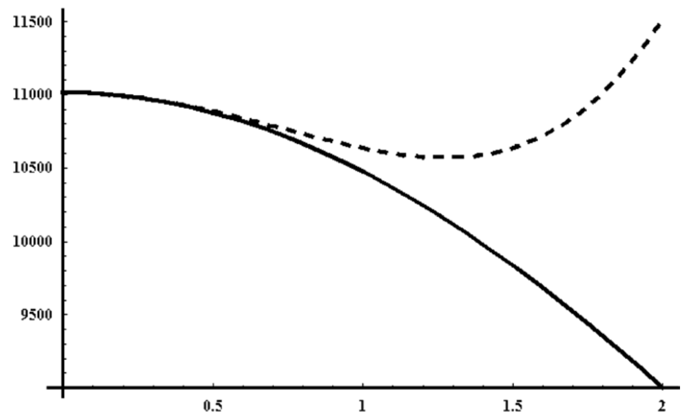

Figure 4. Comparison between the exact (dashed line) and approximate (continuous line) solutions of the ODE (4), i.e. there is a good agreement between the exact and numerical solution within the given range ($r = 0.01$–0.5), which also presented by the error function in figure 4 in the same range as above.

- In the vicinity of the cylinder wall, and close to the region, where the boundary layer phenomena are enhanced (which are characterized by high values of the velocity gradients in direction perpendicular to the groundflow [3]), but still far enough from the boundary layer itself (where the previously mentioned velocity gradients have very large values), the coefficients $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}$ take small values due to $r \approx R$ and (at finite values of the relevant first-, second- and third-order derivatives) may lead to negligibly small summarized values of the last three terms on the left-hand side of equation (4).

These assumptions allow abandoning the last three terms on the left-hand side of (4). Within the frame of the approximation applied, the MAPLE software package [11] gave us a simple analytical new result for the velocity function in the vertical cylindrical tube as is shown below:

$$v(r) = \cosh(A \cdot r) + K_1 \cdot \cosh(A \cdot \sqrt{K - r^2}) + K_2 \cdot \cos(A \cdot r) + K_3 \cdot \cos(A \cdot \sqrt{K - r^2}), \tag{8}$$

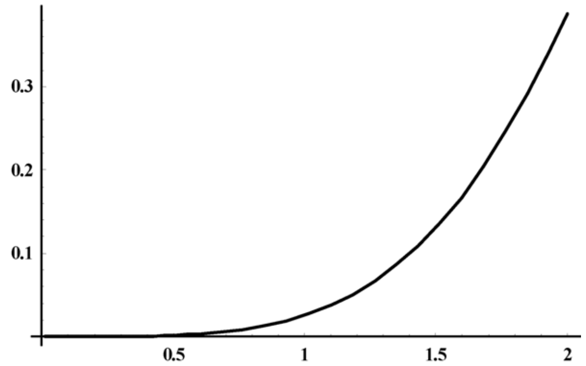

Figure 5. Fitting error in the range ($r = 0.01$–2). (The calculation of error was the next: abs $(F(r) - f(r)/f(r))$, where F = exact and f = approximate functions.)

where K_1, K_2, K_3 and K are the integration constants, while the quantity A is a coefficient emanating from the coefficient in the linear equation (2) and is defined as $A = \sqrt[4]{a}$. This result can easily be obtained without using any software. Then, formula (8) of the velocity function can also be used for the calculation of the explicit form of the temperature perturbation expression within the framework of applied linear approximation [3]. With a view to perform the necessary subsequent calculations, the trigonometric, and hyperbolic functions having in their arguments the expression $A\sqrt{K - r^2}$ will be applied in equation (8) in the usual series expansion forms, i.e.,

$$\cos(A\sqrt{K - r^2}) = \sum_{n=0}^{+\infty} (-1)^n \frac{A^{2n}}{(2n)!}(K - r^2)^n, \qquad \cosh(A\sqrt{K - r^2}) = \sum_{n=0}^{+\infty} \frac{A^{2n}}{(2n)!}(K - r^2)^n. \tag{9}$$

On the basis of (8), the velocity function can be drawn to present the final result graphically (figure 3).

The velocity curve is presented in figure 3. It has an extremal (minimum) value and agrees qualitatively with the velocity function profiles presented in [8] for cross sections of the tubes with an elliptic and circle shape (our result presented in figure 3 corresponds to a half-space of the fully developed buoyancy-induced flow in McBain's paper (p 373)).

The comparison of symbolic/analytical solutions for the complete ODE (4) and its asymptotically simplified form is given in figure 4.

It is clear from figure 5 that there is a very good agreement between the exact and approximate solution till value 0.5 of the independent variable.

To obtain the full description of the convective flow problem examined, it is necessary to solve the differential equation (3) and derive the expression for the temperature disturbance τ too. The double integration of the general term in the series expansions (9) (by MAPLE) leads to the appearance of hypergeometric functions in the series expansions, i.e. from equation (3) we obtained directly

$$\tau = -\frac{C}{\chi}\left(\frac{\cosh(A \cdot r)}{A^2} - K_2 \cdot \frac{\cos(A \cdot r)}{A^2} + K \sum_{n=0}^{+\infty} [K_1 + (-1)^n K_3] \frac{A^{2n}}{(2n)!} \cdot \frac{F\left(\left[-\frac{1}{2}, -n-1\right], \left[\frac{1}{2}\right], \frac{r^2}{K}\right)}{2(n+1)}\right). \tag{10}$$

In case of cylinders with perfectly insulating walls the velocity $\vec{v}$ and the temperature perturbation τ follow the next Dirichlet-, and Neumann-type boundary conditions, respectively:

$$v(R) = 0, \tag{11a}$$

$$\left(\frac{\partial \tau}{\partial r}\right)_{r=R} = 0. \tag{11b}$$

These boundary conditions lead to the reduction of the number of unknown integration constants. However, firstly we exploit the meaning of the velocity function on the axis of the vertical cylindrical tube $v(0)$, i.e.,

$$v(0) = ch(0) + K_1 \cosh(A\sqrt{K}) + K_2 + K_3 \cos(A\sqrt{K}), \tag{12}$$

from which the integration constant K_2 was immediately derived:

$$K_2 = -K_1 \cosh(A\sqrt{K}) - K_3 \cos(A\sqrt{K}), \tag{13}$$

because $v(0)$ was taken here as the unit value of velocity. Application of the Dirichlet-type boundary condition (11a) imposed on the velocity function gives directly

$$K_1 \cdot \left[\cosh(A\sqrt{K - R^2}) - \cosh(A\sqrt{K}) \cos(AR)\right] \\ + K_3 \cdot \left[\cos(A\sqrt{K - R^2}) - \cos(A\sqrt{K}) \cos(AR)\right] + \cosh(AR) = 0. \tag{14}$$

For the relevant calculation of the Neumann-type boundary condition (11b), the following relation for hypergeometric functions was used [14]:

$$\frac{\mathrm{d}}{\mathrm{d}x} F(c; d; e; x) = \frac{cd}{e} F(c + 1; d + 1; e + 1; x), \tag{15}$$

which led to the expression of

$$\left(\frac{\partial \tau}{\partial r}\right)_{r=R} \equiv -\sinh(AR) + K_2 \sin(AR) \\ + AR \sum_{n=0}^{+\infty} [K_1 + (-1)^n K_3] \cdot \frac{A^{2n}}{(2n)!} F\left(\left[\frac{1}{2}, -n\right], \left[\frac{3}{2}\right], \frac{R^2}{K}\right) = 0. \tag{16}$$

Finally, from equations (14) and (16) we derived formulae

$$K_1 = \frac{\cosh(AR) - \frac{\beta_2}{\beta_1} \sinh(AR)}{\alpha_2 - \frac{\beta_2}{\beta_1} \alpha_1}, \qquad K_3 = \frac{\sinh(AR)}{\beta_1} - \frac{\alpha_1 \left[\cosh(AR) - \frac{\beta_2}{\beta_1} \sinh(AR)\right]}{\alpha_2 \beta_1 - \beta_2 \alpha_1}, \tag{17}$$

where the following abbreviations were used:

$$\begin{aligned} \alpha_1 &= AR \sum_{n=0}^{+\infty} \frac{A^{2n}}{(2n)!} F\left(\left[\frac{1}{2}, -n\right], \left[\frac{3}{2}\right], \frac{R^2}{K}\right) - \cosh(A\sqrt{K}) \cdot \sin(AR), \\ \beta_1 &= AR \sum_{n=0}^{+\infty} (-1)^n \frac{A^{2n}}{(2n)!} F\left(\left[\frac{1}{2}, -n\right], \left[\frac{3}{2}\right], \frac{R^2}{K}\right) - \cos(A\sqrt{K}) \cdot \sin(AR), \\ \alpha_2 &= \cosh(A\sqrt{K}) \cos(AR) - \cosh(A\sqrt{K - R^2}), \\ \beta_2 &= \cos(A\sqrt{K}) \cos(AR) - \cos(A\sqrt{K - R^2}). \end{aligned} \tag{18}$$

An unequivocal connection can be established between the integration constants K_1, K_3 and K; K_2 can also be expressed by them using equation (10), and all these relations include the still unknown fourth integration constant K.

A further limit situation is represented by the cylinder surrounded by superficies made from ideally perfect heat conducting material. This case is characterized by the Dirichlet-type boundary condition $\tau(R) = 0$, which is relevant for temperature fluctuation τ [3]. This condition describes the case of 'instantaneous' heat absorption by the cylinder wall.

4. Conclusions

The fourth-order ordinary differential equation describing radial dependence of the velocity function of convective flow in the cross section perpendicular to the axis of the vertical cylinder is studied by the use of advanced symbolic calculation software packages. The complete initial ordinary differential equation of fourth order originating from the Boussinesq system of equations is solved directly and the solution is presented by two formulae via Meijer-, Bessel- and Neumann-type special functions. The relevant solution of the simplified linearized ordinary differential equation is given by simple cosine and hyperbolic cosine functions allowing relatively the simple study of the imposed Dirichlet-, and Neumann-type boundary conditions. The boundary conditions corresponding to the case of cylindrical tubes with perfectly insulating walls made possible the direct determination of three integration constants and only the remaining fourth one is taken as a parameter. As a direct continuation of this work, application of the asymptotic series expansion techniques [16] to the special functions appearing in solutions (5) and (7) may trace out new and fruitful research direction in the general topic of convection instability problems.

Acknowledgments

Cs M acknowledges support of the Bolyai Fellowship BO/00192/05—2005; Á B expresses her thanks to National Office for Research and Technology (Öveges József Fellowship (OMFB-01436/2006)). R Cs acknowledges support of the National Scientific Foundation of Hungary (T047132). The authors wish to thank Dr G Gyulai, Dr K Gottschalk and Dr J Tóth for their comments to the manuscript.

Appendix: the explicit form of the temperature gradient function

The one-time integration of the velocity function (7) with respect to the spatial coordinate r leads to the following result:

$$\begin{aligned}\int v\,\mathrm{d}r = {} & C_1' \cdot \frac{r}{2}\{2J_0(\sqrt[4]{a}r) + \pi H_{s0}(\sqrt[4]{a}r)J_1(\sqrt[4]{a}r) - \pi H_{s1}(\sqrt[4]{a}r)J_0(\sqrt[4]{a}r)\} \\ & + C_2' \cdot \frac{r}{2}\{2Y_0(\sqrt[4]{a}r) + \pi H_{s0}(\sqrt[4]{a}r)Y_1(\sqrt[4]{a}r) - \pi H_{s1}(\sqrt[4]{a}r)Y_0(\sqrt[4]{a}r)\} \\ & + C_3' \cdot \frac{r}{2}\{2J_0(\mathrm{i}\sqrt[4]{a}r) + \pi H_{s0}(\mathrm{i}\sqrt[4]{a}r)J_1(\mathrm{i}\sqrt[4]{a}r) - \pi H_{s1}(\mathrm{i}\sqrt[4]{a}r)J_0(\mathrm{i}\sqrt[4]{a}r)\} \\ & + C_4' \cdot \frac{r}{2}\{2Y_0(\mathrm{i}\sqrt[4]{a}r) + \pi H_{s0}(\mathrm{i}\sqrt[4]{a}r)Y_1(\mathrm{i}\sqrt[4]{a}r) - \pi H_{s1}(\mathrm{i}\sqrt[4]{a}r)Y_0(\mathrm{i}\sqrt[4]{a}r)\},\end{aligned} \tag{A.1}$$

where the functions denoted by $H_{s0}(\sqrt[4]{a}r)$, $H_{s1}(\sqrt[4]{a}r)$, $H_{s0}(\mathrm{i}\sqrt[4]{a}r)$, $H_{s1}(\mathrm{i}\sqrt[4]{a}r)$ are the so-called Struve functions, representing the solution functions of an inhomogeneous Bessel-type ODE [15].

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